

MEASURES AND DIRICHLET FORMS UNDER THE GELFAND TRANSFORM

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Dedicated with deep respect to Professor Ildar Ibragimov on the occasion of his eightieth birthday.

ABSTRACT. Using the standard tools of Daniell-Stone integrals, Stone-Čech compactification and Gelfand transform, we show explicitly that any closed Dirichlet form defined on a measurable space can be transformed into a regular Dirichlet form on a locally compact space. This implies existence, on the Stone-Čech compactification, of the associated Hunt process. As an application, we show that for any separable resistance form in the sense of Kigami there exists an associated Markov process.

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1. INTRODUCTION

The main object of our study is a Dirichlet forms $(\mathcal{E}, \mathcal{F})$ on the L_2 -space over a measure space (X, \mathcal{X}, μ) . The notion of the Dirichlet

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form means that \mathcal{E} is a closed nonnegative (bilinear) quadratic form on $L^2(X, \mathcal{X}, \mu)$ with a dense domain $\mathcal{F} \subset L^2(X, \mathcal{X}, \mu)$. Moreover $(\mathcal{E}, \mathcal{F})$ has what is called Markov (or positivity preserving, or normal contraction property): if $u \in \mathcal{F}$ then $\bar{u} = \min(u, 1) \in \mathcal{F}$ and

$$\mathcal{E}(\bar{u}, \bar{u}) \leq \mathcal{E}(u, u).$$

By the combination of the standard theories of quadratic forms on Hilbert spaces, the spectral theory of self-adjoint operators and the Hille-Yosida theorem, there exists an associated self-adjoint operator (non-negative or non-positive, depending on the analytic or probabilistic conventions), which generates a positivity preserving contraction semigroup on $L^2(X, \mathcal{X}, \mu)$. This is equivalent to having a semigroup of transition probability kernels which, by the Kolmogorov's general theory of random process, is equivalent to the existence of a symmetric Markov process (in the usual way one may have to allow for the extinction of the process, or to augment the state space X with a "cemetery" point). This set up has generated an abundance of strong and well-known results, see e.g. [8, 10, 11, 14, 25, 26, 27], and recently was extensively used in analysis and probability on fractals, see [20, 28, 23]. However most of the basic results in the theory of Dirichlet forms and Markov processes rely on a set up where X is assumed to be a topological space. Examples include the classical Beurling-Deny decomposition for regular Dirichlet forms, the existence of energy measures in the sense of Fukushima [14] and LeJan [24] or the existence of an associated Hunt process. To discuss them most references require X to be locally compact. Of course it is desirable to have versions of these theorems in more general situations (for instance for quasi-regular Dirichlet forms on Souslin spaces), and therefore a reduction of topological assumptions was one of the various directions into which the standard theory for regular Dirichlet forms has been extended. See e.g. [2, 3, 13, 25] for results close to the content of our paper. One of the typical strategies is to embed the possibly non-locally compact state space X into a larger but (locally) compact space and to transfer the Dirichlet form to this new space, where the standard theory for the locally compact case applies. In [1] the authors used such a compactification method to prove a Beurling-Deny type theorem for quasi-regular Dirichlet forms on Hausdorff spaces X that are such that each compact is metrizable and its Borel σ -algebra is countably generated.

In this paper we pursue similar ideas but intend to emphasize a more algebraic point of view and do not assume the given state space X to carry any topology (except for Section 7). Given a multiplicative Stonean vector lattice \mathcal{B} of bounded real-valued functions on a set X we

use the connection between the Daniell-Stone theorem and Gelfand's representation theorem for C^* -algebras to establish an injection of a suitable class of measures on X into the space of nonnegative Radon measures on the spectrum Δ of the complex uniform closure of \mathcal{B} . We apply this idea to show that for any given Dirichlet form over a measurable space there is a corresponding uniquely determined regular Dirichlet form on a larger and locally compact state space.

We consider the algebra $\mathcal{B}(\mathcal{E})$ of bounded measurable functions on (X, \mathcal{X}) that are μ -square integrable and have finite energy. The uniform closure of its complexification is a C^* -algebra, and its spectrum Δ is a locally compact Hausdorff space. If $\mathcal{B}(\mathcal{E})$ vanishes nowhere, then Δ (roughly speaking) contains X as a dense subset, and there is a Radon measure $\hat{\mu}$ on Δ which is uniquely determined by μ in a way that makes the restriction of the Gelfand transform $f \mapsto \hat{f}$ to $\mathcal{B}(\mathcal{E})$ an L_2 -isometry, i.e.

$$(1) \quad \|\hat{f}\|_{L_2(\Delta, \hat{\mu})} = \|f\|_{L_2(X, \mu)}, \quad f \in \mathcal{B}(\mathcal{E}).$$

This allows to define a symmetric bilinear form by

$$\hat{\mathcal{E}}(\hat{f}, \hat{g}) := \mathcal{E}(f, g), \quad f, g \in \mathcal{B}(\mathcal{E}).$$

Our main result, Theorem 5.1, says that $\hat{\mathcal{E}}$, together with the image $\hat{\mathcal{B}}(\mathcal{E})$ of $\mathcal{B}(\mathcal{E})$ under the Gelfand map, is closable, and its closure $(\hat{\mathcal{E}}, \hat{\mathcal{F}})$ in $L_2(\Delta, \hat{\mu})$ is a symmetric regular Dirichlet form. In other words, we can find a locally compact Hausdorff space Δ which 'contains' the state space X , and a regular Dirichlet form $(\hat{\mathcal{E}}, \hat{\mathcal{F}})$ that is the image of $(\mathcal{E}, \mathcal{F})$. For this Dirichlet form we can now apply the standard theory [14] and for instance obtain a Beurling-Deny representation and the existence of energy measures. We would like to point out that in [2] the embedding of a Souslin standard Borel space into the Gelfand spectrum of a countably generated and point separating algebra of continuous functions had been used to construct a symmetric Hunt process associated with the given Dirichlet form.

In contrast to references like [1, 2] it may not be possible to pull these results back to the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on the original state space X . For instance, the energy measure of $(\hat{\mathcal{E}}, \hat{\mathcal{F}})$ on Δ may be such that the image of X under the embedding into Δ is of zero energy measure, see Example 6.1. This is reminiscent of the situation in infinite dimensional analysis where the Cameron-Martin space typically is a null set, cf. Remark 6.2 and such references as [16, 17, 18, 29]. The study of the Dirichlet form $(\hat{\mathcal{E}}, \hat{\mathcal{F}})$ on the spectrum Δ may be a natural

way to enlarge the space to support energy measures. Under additional topological assumptions we can recover results similar to those in [1, 2].

Before we turn to Dirichlet forms we discuss how to naturally relate suitable measures μ on X to Radon measures $\hat{\mu}$ on Δ . This correspondence relies on a connection between the Daniell-Stone theorem and the Gelfand transform. Although this idea is not new, see for instance [15], it does not seem to be all too well known. We consider a multiplicative vector lattice \mathcal{B} of bounded real-valued functions on X . The uniform closure $A(\mathcal{B})$ of its complexification is a commutative C^* -algebra. If μ is uniquely associated with a positive linear functional on \mathcal{B} then we may use positivity arguments to obtain a uniquely associated positive linear functional on the space $C_c(\Delta, \mathbb{R})$ of real-valued compactly supported functions on the spectrum of $A(\mathcal{B})$. By the Riesz representation theorem this functional can be represented by integration with respect to some uniquely determined Radon measure $\hat{\mu}$ on Δ . Proceeding this way we obtain an injective mapping from a cone of nonnegative measures on X into the cone of nonnegative Radon measures on Δ . The isomorphism property of the Gelfand transform finally yields the L_2 -isometry (1).

The paper is organized as follows. For convenience, we recall some preliminaries concerning Gelfand theory and the Daniell-Stone theorem in the section 2. In Section 3 we investigate the connection for multiplicative Stonean vector lattices of bounded real-valued functions and establish some lemmas on positivity, support properties and denseness. The main result of Section 4 is Theorem 4.1, which states the correspondence between measures on X and Δ . As a consequence we also obtain the L_2 -isometry (1). In Section 5 we apply these results to Dirichlet forms to obtain the closability of $(\hat{\mathcal{E}}, \hat{\mathcal{B}}(\mathcal{E}))$ in $L_2(\Delta, \hat{\mu})$ and the regularity of its closure $(\hat{\mathcal{E}}, \hat{\mathcal{F}})$, Theorem 5.1. Consequences include the Beurling-Deny representation and the existence of Radon energy measures for the transferred Dirichlet form $(\hat{\mathcal{E}}, \hat{\mathcal{F}})$ on Δ , sketched in Section 6.

We write $C_0(\Delta)$ to denote the space of continuous functions on Δ that vanish at infinity and $C_c(\Delta)$ to denote its subspace of functions with compact support. For their subspaces of real-valued functions we write $C_0(\Delta, \mathbb{R})$ and $C_c(\Delta, \mathbb{R})$, respectively, and we will do similarly for other function spaces. If the index set of a sequence is not specified, it is the set of natural numbers, and if corresponding limits are taken, they are taken with the index going to infinity.

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2. GELFAND THEORY AND THE DANIELL-STONE THEOREM

For multiplicative vector lattices of bounded real valued functions the theorem of Daniell-Stone can be connected to Gelfand's representation theorem for commutative C^* -algebras. In this section we briefly recall these two concepts.

We start with remarks on *commutative Gelfand theory*, cf. [5, 7, 19]. Let A be a commutative C^* -algebra of bounded functions $a : X \rightarrow \mathbb{C}$, with the supremum norm $\|\cdot\|$ and with the algebra operations defined pointwise and the involution $*$ defined by complex conjugation $a^* := \bar{a}$. By $\Delta(A)$ we denote the *spectrum (Gelfand space)* of A , the space of continuous, complex-valued, multiplicative functionals on A . Equipped with the Gelfand topology the spectrum $\Delta(A)$ becomes a regular locally compact Hausdorff space, cf. [19]. If A contains the constant function $\mathbf{1}$ then $\Delta(A)$ is compact. The space $\Delta(A)$ is second countable if and only if the C^* -algebra A is separable, and this in turn is equivalent to A being countably generated. For any $a \in A$ the *Gelfand transform* $\hat{a} : \Delta(A) \rightarrow \mathbb{C}$ of a is defined by $\hat{a}(\varphi) := \varphi(a)$, and by the Gelfand representation theorem the Gelfand map $a \mapsto \hat{a}$ is seen to be an isometric $*$ -isomorphism from the Banach algebra A onto the algebra $C_0(\Delta(A))$ of continuous functions on $\Delta(A)$ vanishing at infinity. If the algebra A vanishes nowhere on X , that is, if for any $x \in X$ there exists some $a \in A$ such that $a(x) \neq 0$, then X may be identified with a subset of $\Delta(A)$ by the map $\iota : X \rightarrow \Delta(A)$, where

$$(2) \quad \iota(x)(a) := a(x) \ , \quad a \in A,$$

for any $x \in X$. Note that multiplication in $C_0(\Delta(A))$ is given pointwise, and

$$\iota(x)(a_1 a_2) = (a_1 a_2)(x) = a_1(x) a_2(x) = \iota(x)(a_1) \iota(x)(a_2)$$

for any $x \in X$ and $a_1, a_2 \in A$. Thus, we observe the set-theoretic inclusion $\iota(X) \subset \Delta(A)$. The set $\iota(X)$ is dense in $\Delta(A)$. For if not, we could find a nonzero function $f \in C_0(\Delta(A))$ such that $f(\iota(x)) = 0$ for all $x \in X$. Then, however, some nonzero $a \in A$ would have to exist with $\hat{a} = f \in C_0(\Delta(A))$, hence

$$(3) \quad \hat{a}(\iota(x)) = \iota(x)(a) = a(x)$$

would have to be zero for all $x \in X$ and consequently $a \equiv 0$ in A , a contradiction.

The second tool we would like to sketch is the *Daniell-Stone Theorem*. Let $X \neq \emptyset$ and let \mathcal{L} be a real vector lattice of functions on X , i.e. a vector space of functions $f : X \rightarrow \mathbb{R}$ that is closed under minimum and maximum operations $f \wedge g = \min(f, g)$ and $f \vee g = \max(f, g)$. We

assume that \mathcal{L} possesses the *Stone property*: for any $f \in \mathcal{L}$, $f \wedge 1 \in \mathcal{L}$. By $\sigma(\mathcal{L})$ we denote the σ -ring of subsets of X generated by \mathcal{L} and by $\mathcal{M}^+(\sigma(\mathcal{L}))$, the cone of (nonnegative) measures on $\sigma(\mathcal{L})$. A positive linear functional $I : \mathcal{L} \rightarrow \mathbb{R}$ is called a *Daniell integral* on \mathcal{L} if for any sequence $(f_n)_n \subset \mathcal{L}$ of nonnegative functions decreasing to zero pointwise at all $x \in X$ also the sequence of integrals $(I(f_n))_n$ decreases to zero. The Daniell-Stone Theorem says that for any Daniell integral I on \mathcal{L} there exists a uniquely determined measure $\mu \in \mathcal{M}^+(\sigma(\mathcal{L}))$ on $\sigma(\mathcal{L})$ such that

$$(4) \quad I(f) = \int_X f d\mu, \quad f \in \mathcal{L}.$$

See for instance [12]. We use the notation

$$\mathcal{D}(\mathcal{L}) := \{ \mu \in \mathcal{M}^+(\sigma(\mathcal{L})) : \text{all functions from } \mathcal{L} \text{ are } \mu\text{-integrable} \}.$$

If I is a Daniell integral on \mathcal{L} then the measure μ uniquely associated with I by (4) is a member of $\mathcal{D}(\mathcal{L})$. Conversely any $\mu \in \mathcal{D}(\mathcal{L})$ defines a Daniell integral on \mathcal{L} by (4). Note that if \mathcal{L} contains a strictly positive function, then all measures in $\mathcal{D}(\mathcal{L})$ are σ -finite, and if it contains the constant function $\mathbf{1}$, then all measures in $\mathcal{D}(\mathcal{L})$ are finite.

3. MULTIPLICATIVE STONEAN VECTOR LATTICES

We are interested in special cases to which both theories apply. Let \mathcal{B} be a real *multiplicative* vector lattice of *bounded* functions on $X \neq \emptyset$ that has the Stone property. By $\mathcal{B} + i\mathcal{B}$ we denote its complexification, that is the complex vector space of functions $f_1 + if_2$ with $f_1, f_2 \in \mathcal{B}$. The vector space operations and the complex conjugation are defined pointwise. We endow $\mathcal{B} + i\mathcal{B}$ with the supremum norm $\|\cdot\|$ and denote its closure by $A(\mathcal{B})$, clearly a Banach space. Pointwise multiplication turns $A(\mathcal{B})$ into a commutative Banach algebra, and with the involution $*$ defined by complex conjugation it becomes a commutative C^* -algebra. Under the Gelfand transform $f \mapsto \hat{f}$ the C^* -algebra $A(\mathcal{B})$ is isometrically $*$ -isomorphic to $C_0(\Delta(A(\mathcal{B})))$. To shorten notation we will write Δ to abbreviate $\Delta(A(\mathcal{B}))$. From now on we will assume the following.

Assumption 3.1. The space \mathcal{B} vanishes nowhere.

Under this assumption the set $\iota(X)$, where ι is defined as in (2) with $A = A(\mathcal{B})$, is a dense subset of Δ , and according to (3) we have $\hat{f}(\iota(x)) = f(x)$ for any $f \in \mathcal{B}$ and $x \in X$.

To discuss nonnegativity issues let $A(\mathcal{B})^+$ and $C_0(\Delta)^+$ denote the cones of real-valued nonnegative functions in $A(\mathcal{B})$ and $C_0(\Delta)$, respectively. For a real-valued function f we write $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$. If f is a member of \mathcal{B} then so are f^+ and f^- .

Lemma 3.1. *A function $f \in A(\mathcal{B})$ is real-valued if and only if $\hat{f} \in C_0(\Delta)$ is. Moreover, we have $f \in A(\mathcal{B})^+$ if and only if $\hat{f} \in C_0(\Delta)^+$.*

This lemma is a consequence of (3) together with the denseness of $\iota(X)$ in Δ .

Lemma 3.2. *For any real-valued $f \in A(\mathcal{B})$ we have $(f^+)^{\wedge} = \hat{f}^+$ and $(f^-)^{\wedge} = \hat{f}^-$.*

Proof. For any $x \in X$ we have $(f^+)^{\wedge}(\iota(x)) = f^+(x)$ by (3). If $f(x) \geq 0$ then $f^+(x) = f(x) = \hat{f}(\iota(x)) = \hat{f}^+(\iota(x))$. If $f(x) < 0$ then $\hat{f}(\iota(x)) < 0$ and $\hat{f}^+(\iota(x)) = 0$. Consequently $(f^+)^{\wedge}(\iota(x)) = \hat{f}^+(\iota(x))$ for all $x \in X$, and by linearity also $(f^-)^{\wedge}(\iota(x)) = \hat{f}^-(\iota(x))$. By continuity and the denseness of $\iota(X)$ in Δ the lemma follows. \square

The members of $A(\mathcal{B})^+$ are all monotone limits of nonnegative functions from \mathcal{B} . For this statement Assumption 3.1 is not needed.

Lemma 3.3. *For any function $f \in A(\mathcal{B})^+$ there exists a monotonically increasing sequence $(f_n)_n$ of nonnegative functions $f_n \in \mathcal{B}$ that converges to f pointwise.*

Proof. By the lattice property in \mathcal{B} , we can see that there is a sequence $(g_n)_n$ of nonnegative functions $g_n \in \mathcal{B}$ converging uniformly to f . We may assume that the nonnegative numbers $\delta_n := \sup_X |g_n - g_{n+1}|$ are such that $\sum_n \delta_n < \infty$ (otherwise pass to a subsequence). Setting $f_n := g_n - g_n \wedge (\sum_{k=n}^{\infty} \delta_k)$ we obtain a sequence $(f_n)_n$ with the desired properties. \square

We discuss compactly supported functions. If \mathcal{B} contains the constant functions, then Δ is compact, hence every function in $\hat{\mathcal{B}}$ has compact support. To formulate a result for the general case, set

$$\mathcal{B}_c := \{\varphi \in \mathcal{B} : \hat{\varphi} \in C_c(\Delta)\}.$$

Clearly \mathcal{B}_c is again a multiplicative vector lattice having the Stone property.

Lemma 3.4. *The space \mathcal{B}_c is uniformly dense in \mathcal{B} .*

To prove Lemma 3.4 we use a property of upper level sets. Given $\varphi \in \mathcal{B}$ and $k \in \mathbb{N} \setminus \{0\}$ set

$$N_k(f) := \left\{ x \in X : |f(x)| \geq \frac{1}{k} \right\}.$$

Lemma 3.5. *For any $f \in \mathcal{B}$ and any k the closure of the set $\iota(N_k(f))$ is compact in Δ .*

Proof. We have $|f| \in \mathcal{B}$ and, according to Lemma 3.2, $|f|^\wedge = |\hat{f}|$. Consequently we may assume $f \geq 0$. Since $\hat{f} \in C_0(\Delta)$, the closed set

$$L_k(f) := \left\{ y \in \Delta : \hat{f}(y) \geq \frac{1}{k} \right\}$$

is contained in a compact set and therefore compact itself. On the other hand $\iota(N_k(f)) \subset L_k(f)$, what implies that $\overline{\iota(N_k(f))}$ is a closed subset of $L_k(f)$, hence compact. \square

We verify Lemma 3.4.

Proof. It suffices to show that nonnegative functions can be approximated. Given $f \in \mathcal{B}$ with $f \geq 0$ consider the functions

$$\varphi_k := f - f \wedge \frac{1}{k}.$$

Obviously the sequence $(\varphi_k)_k$ uniformly converges to f , and for fixed k the set

$$N^k := \{x \in X : \varphi_k(x) > 0\}$$

is a subset of $N_k(f)$. On the other hand, we have

$$\{y \in \Delta : \hat{\varphi}_k(y) > 0\} \subset \overline{\iota(N^k)}.$$

For if there were some $y \in \Delta$ with $\hat{\varphi}_k(y) > 0$ having an open neighborhood U_y such that $\varphi_k(x) = 0$ for all $x \in X$ with $\iota(x) \in U_y$, then we would have $\hat{\varphi}_k(z) = 0$ for all $z \in U_y$ by the density of $\iota(X)$ in Δ , a contradiction. It also follows that

$$\text{supp } \hat{\varphi}_k \subset \overline{\iota(N^k)} \subset \overline{N_k(f)},$$

and Lemma 3.5 implies that $\text{supp } \hat{\varphi}_k$ is compact. \square

4. POSITIVE LINEAR FUNCTIONALS AND MEASURES

In this section we establish a correspondence between suitable measures μ on X and Radon measures $\hat{\mu}$ on Δ and list some consequences. As before we assume that \mathcal{B} is a Stonean multiplicative vector lattice of bounded real-valued functions on X .

Let $I : \mathcal{B} \rightarrow \mathbb{R}$ be a positive linear functional. Given a function $f \in A(\mathcal{B})^+$ and an increasing sequence $(f_n)_n \subset \mathcal{B}$ of nonnegative function as in Lemma 3.3, we set

$$(5) \quad I(f) := \sup_n I(f_n).$$

The lattice property of \mathcal{B} guarantees that (5) provides a well-defined positive linear (i.e. positively homogeneous and additive) functional $I : A(\mathcal{B})^+ \rightarrow [0, +\infty]$. In what follows let Assumption 3.1 be satisfied. In view of Lemma 3.1 we can then define a bounded positive linear functional $\hat{I} : C_0(\Delta)^+ \rightarrow [0, +\infty]$ by

$$(6) \quad \hat{I}(\hat{f}) := I(f), \quad \hat{f} \in C_0(\Delta)^+,$$

and according to Lemma 3.2 we may set $I(f) := I(f^+) - I(f^-)$ and $\hat{I}(\hat{f}) := I(f)$ to extend (6) to all $f \in \mathcal{B}$.

Let $\mathcal{M}^+(\Delta)$ denote the cone of nonnegative Radon measures on Δ . The Riesz representation theorem ensures the existence of a uniquely determined $\hat{\mu} \in \mathcal{M}^+(\Delta)$ such that for any $\hat{f} \in C_c(\Delta)$ we have

$$(7) \quad \hat{I}(\hat{f}) = \int_{\Delta} \hat{f} d\hat{\mu}.$$

Remark 4.1. 2 Recall that to prove the existence part of the Riesz representation theorem one usually sets

$$\hat{\mu}(K) := \inf \left\{ \hat{I}(\hat{f}) : f \in C_c(\Delta, \mathbb{R}), \text{ and } f \geq \mathbf{1}_K \right\}$$

for compact $K \subset \Delta$ and defines the $\hat{\mu}$ -measure of an arbitrary Borel set by inner approximation by compacts. It is therefore sufficient to know the functional \hat{I} on the cone $C_c(\Delta)^+$.

Now assume that $I : \mathcal{B} \rightarrow \mathbb{R}$ is a Daniell integral and $\mu \in \mathcal{D}(\mathcal{B})$ is the unique measure on $\sigma(\mathcal{B})$ associated with I as in (4). In this case definition (6) yields

$$(8) \quad \int_X f d\mu = \int_{\Delta} \hat{f} d\hat{\mu}, \quad f \in \mathcal{B}.$$

The map $\mu \mapsto \hat{\mu}$ is positive and linear (i.e. additive and positively homogeneous). By (8) and the uniqueness part of the Daniell-Stone Theorem we obtain the following result.

Theorem 4.1. *The map $\mu \mapsto \hat{\mu}$ is an injection of $\mathcal{D}(\mathcal{B})$ into $\mathcal{M}^+(\Delta)$.*

We may also consider equivalence classes of functions.

Lemma 4.1. *Let $\mu \in \mathcal{D}(\mathcal{B})$ and $f \in \mathcal{B}$. Then $f = 0$ μ -a.e. on X if and only if $\hat{f} = 0$ $\hat{\mu}$ -a.e. on Δ .*

Proof. Let $f = 0$ μ -a.e. on X . Then also f^+ and f^- vanish μ -a.e. on X . By Lemma 3.2 and (8) therefore $\int_{\Delta} \hat{f}^+ d\hat{\mu} = 0$, hence $\hat{f}^+ = 0$ $\hat{\mu}$ -a.e. The same is true for \hat{f}^- and consequently $\hat{f} = 0$ $\hat{\mu}$ -a.e. The converse implication follows in a similar manner. \square

Therefore the Gelfand map induces a well-defined map from the space of μ -equivalence classes of functions from \mathcal{B} into the space of $\hat{\mu}$ -equivalence classes of functions on Δ . We denote it again by $f \mapsto \hat{f}$. We investigate corresponding L_2 -spaces.

Lemma 4.2. *Let $\mu \in \mathcal{D}(\mathcal{B})$. For $f \in \mathcal{B}$ we have*

$$\|f\|_{L_2(X, \mu)} = \|\hat{f}\|_{L_2(\Delta, \hat{\mu})}.$$

Proof. Being an algebra homomorphism, the Gelfand map satisfies $(\hat{f})^2 = (f^2)^\wedge$ for any $f \in A(\mathcal{B})$. For $f \in \mathcal{B}$ the identity (8) then yields

$$\int_X f^2 d\mu = \int_{\Delta} (f^2)^\wedge d\hat{\mu} = \int_{\Delta} \hat{f}^2 d\hat{\mu}.$$

\square

The following fact will be used in the next section.

Lemma 4.3. *For any $\mu \in \mathcal{D}(\mathcal{B})$ the image $\hat{\mathcal{B}}$ of \mathcal{B} is dense in $L_2(\Delta, \hat{\mu}, \mathbb{R})$.*

Proof. Since $C_0(\Delta, \mathbb{R})$ is a dense subspace of $L_2(\Delta, \hat{\mu}, \mathbb{R})$, it suffices to show that any $\hat{f} \in C_0(\Delta, \mathbb{R})$ can be approximated in $L_2(\Delta, \hat{\mu}, \mathbb{R})$ by functions from $\hat{\mathcal{B}}$. However, as $C_0(\Delta)$ is isometrically isomorphic to the uniform closure $A(\mathcal{B})$ of the complexification of \mathcal{B} , there is a sequence $(f_n)_n \subset \mathcal{B}$ such that $(\hat{f}_n)_n$ approximates \hat{f} uniformly. Given $\varepsilon > 0$ we can find a compact set $K_\varepsilon \subset \Delta$ such that $\hat{\mu}(\Delta \setminus K_\varepsilon) < \varepsilon$. Then obviously

$$\lim_n \int_{K_\varepsilon} |\hat{f}_n - \hat{f}|^2 d\hat{\mu} = 0$$

and

$$\int_{\Delta \setminus K_\varepsilon} |\hat{f}_n - \hat{f}|^2 d\hat{\mu} \leq \varepsilon (\|f\| + \sup_n \|f_n\|).$$

\square

5. DIRICHLET FORMS UNDER THE GELFAND MAP

We use the setup of the previous section to transfer from a Dirichlet form on a measure space to a regular Dirichlet form on a locally compact Hausdorff space.

Let (X, \mathcal{X}, μ) be a measure space and $(\mathcal{E}, \mathcal{F})$ a Dirichlet form on $L_2(X, \mu, \mathbb{R})$, see for example [8, Chapter I]. We will frequently use the shorthand notation $\mathcal{E}(f) := \mathcal{E}(f, f)$ and do similarly for other bilinear expressions. The space of bounded measurable functions on X is denoted by $b\mathcal{X}$. Set

$$(9) \quad \mathcal{B}(\mathcal{E}) := \{f \in b\mathcal{X} : \text{the } \mu\text{-equivalence class of } f \text{ is in } \mathcal{F} \cap L_1(X, \mu, \mathbb{R})\}.$$

The Cauchy-Schwarz inequality and the Markov property of $(\mathcal{E}, \mathcal{F})$ imply that $\mathcal{B}(\mathcal{E})$ is a multiplicative vector lattice that has the Stone property. In addition we assume the following:

Assumption 5.1. The space $\mathcal{B}(\mathcal{E})$ vanishes nowhere on X .

Let Δ be the spectrum of the uniform closure $A(\mathcal{B}(\mathcal{E}))$ of the complexification of \mathcal{B} . For $f, g \in \mathcal{B}(\mathcal{E})$ we set

$$(10) \quad \hat{\mathcal{E}}(\hat{f}, \hat{g}) := \mathcal{E}(f, g).$$

Obviously $\hat{\mathcal{E}}$ is a nonnegative definite symmetric bilinear form on the dense subspace

$$\hat{\mathcal{B}}(\mathcal{E}) = \left\{ \hat{f} \in C_0(\Delta, \mathbb{R}) : f \in \mathcal{B}(\mathcal{E}) \right\}$$

of $L_2(\Delta, \hat{\mu}, \mathbb{R})$. It enjoys the Markov property. In fact, it defines a regular symmetric Dirichlet form on $L_2(\Delta, \hat{\mu}, \mathbb{R})$.

Theorem 5.1. *The form $(\hat{\mathcal{E}}, \hat{\mathcal{B}}(\mathcal{E}))$ is closable on $L_2(\Delta, \hat{\mu}, \mathbb{R})$. Its closure $(\hat{\mathcal{E}}, \hat{\mathcal{F}})$ defines a symmetric regular Dirichlet form.*

Proof. Let $(\hat{f}_n)_n$ be a sequence of functions from $\hat{\mathcal{B}}(\mathcal{E})$ that is $\hat{\mathcal{E}}$ -Cauchy and tends to zero in $L_2(\Delta, \hat{\mu}, \mathbb{R})$. Then by (10) the sequence $(f_n)_n$ of preimages $f_n \in \mathcal{B}(\mathcal{E})$ of the functions \hat{f}_n under the Gelfand map is \mathcal{E} -Cauchy, and by Lemma 4.2 it tends to zero in $L_2(X, \mu)$. From the closability of $(\mathcal{E}, \mathcal{F})$ together with (10) it then follows that

$$\lim_n \hat{\mathcal{E}}(\hat{f}_n) = \lim_n \mathcal{E}(f_n) = 0.$$

Therefore $(\hat{\mathcal{E}}, \hat{\mathcal{B}}(\mathcal{E}))$ is closable. According to Lemma 3.4 the set

$$\hat{\mathcal{B}}_c(\mathcal{E}) := \left\{ \hat{f} \in C_c(\Delta) : f \in \mathcal{B}(\mathcal{E}) \right\}$$

is uniformly dense in $\hat{\mathcal{B}}(\mathcal{E})$, hence also in $C_0(\Delta)$. On the other hand, given $f \in \mathcal{B}(\mathcal{E})$, the functions

$$\varphi_k := f - (f \vee (-\frac{1}{k})) \wedge \frac{1}{k}$$

converge to f in \mathcal{E}_1 -norm, see for instance [14, Theorem 1.4.2]. Consequently $\hat{\mathcal{B}}_c(\mathcal{E})$ is a core for $(\hat{\mathcal{E}}, \hat{\mathcal{F}})$. \square

To the symmetric regular Dirichlet form $(\hat{\mathcal{E}}, \hat{\mathcal{F}})$ on $L_2(\Delta, \hat{\mu}, \mathbb{R})$ we refer as the *transferred Dirichlet form*.

6. BEURLING-DENY DECOMPOSITION AND ENERGY MEASURES

We record some consequences of the existing theory for Dirichlet forms on locally compact spaces when applied to $(\hat{\mathcal{E}}, \hat{\mathcal{F}})$. As before let (X, \mathcal{X}, μ) be a measure space and $(\mathcal{E}, \mathcal{F})$ a symmetric Dirichlet form on $L_2(X, \mu)$ such that Assumption 5.1 is satisfied.

The first theorem is the *Beurling-Deny representation*.

Theorem 6.1. *The transferred Dirichlet form $(\hat{\mathcal{E}}, \hat{\mathcal{F}})$ on $L_2(\Delta, \hat{\mu}, \mathbb{R})$ admits the decomposition*

$$\begin{aligned} \hat{\mathcal{E}}(\hat{f}, \hat{g}) &= \hat{\mathcal{E}}^c(\hat{f}, \hat{g}) + \int \int_{\Delta \times \Delta} (\hat{f}(x) - \hat{f}(y))(\hat{g}(x) - \hat{g}(y)) \hat{J}(dx, dy) \\ &\quad + \int_{\Delta} \hat{f}(x) \hat{g}(x) \hat{\kappa}(dx) \end{aligned}$$

for any $\hat{f}, \hat{g} \in \hat{\mathcal{B}}(\mathcal{E})$, where $\hat{\mathcal{E}}^c$ is a symmetric nonnegative definite bilinear form on $\hat{\mathcal{B}}(\mathcal{E})$ that is strongly local, \hat{J} is a symmetric nonnegative Radon measure on $\Delta \times \Delta \setminus \{(x, x) : x \in \Delta\}$, and $\hat{\kappa}$ is a nonnegative Radon measure on Δ . The normal contraction operates on $\hat{\mathcal{E}}^c$, and the triple $(\hat{\mathcal{E}}^c, \hat{J}, \hat{\kappa})$ is uniquely determined.

For a proof see for instance [4] or [14].

Remark 6.1. Note that these proofs require the local compactness but not the second countability of Δ . However, if $\mathcal{B}(\mathcal{E})$ has a countable subset from which any element in $\mathcal{B}(\mathcal{E})$ can be produced by linear operations, multiplication, truncation by 1 and taking uniform limits, then Δ is second countable and by Urysohn's theorem there exists a metric turning Δ into a locally compact separable metric space.

Another result is the *existence of energy measures* for the transferred Dirichlet form $(\hat{\mathcal{E}}, \hat{\mathcal{F}})$, which is an immediate consequence its regularity, [14, 24].

Theorem 6.2. *For any $\hat{f} \in \hat{\mathcal{B}}(\mathcal{E})$ there exists a uniquely determined finite nonnegative Radon measure $\hat{\Gamma}(\hat{f})$ on Δ such that*

$$2 \int_{\Delta} \hat{\varphi} d\hat{\Gamma}(\hat{f}) = 2\hat{\mathcal{E}}(\hat{\varphi}\hat{f}, \hat{f}) - \hat{\mathcal{E}}(\hat{\varphi}, \hat{f}^2)$$

for any $\hat{\varphi} \in \hat{\mathcal{B}}(\mathcal{E})$.

If the original Dirichlet form $(\mathcal{E}, \mathcal{F})$ itself admits energy measures, that is if for any $f \in \mathcal{B}(\mathcal{E})$ there exists some nonnegative measure $\Gamma(f)$ such that

$$(11) \quad 2 \int_X \varphi d\Gamma(f) = 2\mathcal{E}(\varphi f, f) - \mathcal{E}(\varphi, f^2), \quad \varphi \in \mathcal{B}(\mathcal{E}),$$

then the energy measures $\hat{\Gamma}(\hat{f})$ are consistent with these original ones.

Theorem 6.3. *Assume that $(\mathcal{E}, \mathcal{F})$ admits energy measures (11). Then for any $f \in \mathcal{B}(\mathcal{E})$ we have*

$$(\Gamma(f))^\wedge = \hat{\Gamma}(\hat{f}).$$

Proof. For any $\hat{\varphi} \in C_0(\Delta)$ we have

$$\begin{aligned} \int_\Delta \hat{\varphi} d(\Gamma(f))^\wedge &= \int_X \varphi d\Gamma(f) \\ &= 2\mathcal{E}(f\varphi, f) - \mathcal{E}(\varphi, f^2) \\ &= 2\hat{\mathcal{E}}((f\varphi)^\wedge, \hat{f}) - \hat{\mathcal{E}}(\hat{\varphi}, (f^2)^\wedge) \\ &= 2\hat{\mathcal{E}}(\hat{f}\hat{\varphi}, \hat{f}) - \hat{\mathcal{E}}(\hat{\varphi}, \hat{f}^2) \\ &= \int_\Delta \hat{\varphi} d\hat{\Gamma}(\hat{f}). \end{aligned}$$

□

Theorem 6.2 is significant, because as the following examples show, the original Dirichlet form $(\mathcal{E}, \mathcal{F})$ itself may not admit energy measures.

Examples 6.1. Consider the classical Dirichlet integral on the unit interval $[0, 1]$, given by

$$\mathcal{E}_0(g) := \int_0^1 g'(x)^2 dx$$

for any function g from

$$\mathcal{F}_0 := \{g \in C([0, 1]) : \mathcal{E}_0(g) < \infty\}.$$

The form $(\mathcal{E}_0, \mathcal{F}_0)$ is a resistance form on $[0, 1]$ in the sense of Kigami [21, 22]. We consider the countable state space $X = \mathbb{Q} \cap [0, 1]$. Set

$$\mathcal{F}_0|_X := \{f : X \rightarrow \mathbb{R} : \text{there exists some } g \in \mathcal{F}_0 \text{ such that } f = g|_X\}$$

and

$$\mathcal{E}(f) := \mathcal{E}_0(g), \quad f \in \mathcal{F}_0|_X.$$

Here $g|_X$ denotes the pointwise restriction of the continuous function g to X . By continuity and the density of X in $[0, 1]$ each $f \in \mathcal{F}_0|_X$ is

the restriction of exactly one function $g \in \mathcal{F}_0$. Now let δ_q denote the normed Dirac point measure at a given point q and let $\{q_n\}_{n=1}^\infty$ be an enumeration of X . Then

$$\mu := \sum_{n=1}^{\infty} 2^{-n} \delta_{q_n}$$

is a probability measure. The form $(\mathcal{E}, \mathcal{F}_0|_X)$ is closable in $L_2(X, \mu)$, see for instance [22, Lemma 9.2 and Theorem 9.4], and its closure $(\mathcal{E}, \mathcal{F})$ is a Dirichlet form. For a function $f \in \mathcal{F}_0|_X$ with $\mathcal{E}(f) > 0$ (such as for instance the restriction to X of a nonconstant linear function) and $g \in \mathcal{F}_0$ is such that $f = g|_X$ we have

$$(12) \quad 2\mathcal{E}(\varphi f, f) - \mathcal{E}(\varphi, f^2) = 2 \int_0^1 \psi(x) g'(x)^2 dx,$$

for all $\varphi \in \mathcal{F}_0|_X$ with $\varphi = \psi|_X$, $\psi \in \mathcal{F}_0$. On the other hand approximation by piecewise linear functions shows that \mathcal{F}_0 is dense in $C([0, 1])$, and consequently any bounded Borel function on $[0, 1]$ can be approximated pointwise by a uniformly bounded sequence of functions from \mathcal{F}_0 . Let $(\psi_n)_n \subset \mathcal{F}_0$ be a uniformly bounded sequence of functions that approximate $\mathbf{1}_X$ pointwise. If for some f as above $(\mathcal{E}, \mathcal{F})$ would admit energy measures as in (11) then we would obtain

$$\int_X \psi_n|_X d\Gamma(f) = \int_{\Delta} (\psi_n|_X)^\wedge d\hat{\Gamma}(f) = \int_0^1 \psi_n(x) g'(x)^2 dx,$$

and by bounded convergence

$$\mathcal{E}(f) = \Gamma(f)(X) = \int_X g'(x)^2 dx = 0,$$

because the restriction of $g'(x)^2 dx$ to X is the zero measure. This contradicts $\mathcal{E}(f) > 0$.

Remark 6.2. In some sense the situation of Example 6.1 displays a similar feature as we encounter it for Dirichlet forms on infinite dimensional spaces. For instance, let (E, H, μ) be an abstract Wiener space, cf. [8, 16, 17, 25, 29], let

$$\mathcal{FC}_b^\infty := \{f(l_1, \dots, l_n) : n \in \mathbb{N}, f \in C_b^\infty(\mathbb{R}^n), l_1, \dots, l_n \in E'\},$$

$$\langle \nabla u(z), h \rangle_H := \frac{\partial u}{\partial h}(z), \quad h \in H,$$

for any $u \in \mathcal{FC}_b^\infty$ and

$$\mathcal{E}(u) := \int_E \|\nabla u\|_H^2 d\mu.$$

Then $(\mathcal{E}, \mathcal{FC}_b^\infty)$ is closable on $L_2(E, \mu)$ and its closure $(\mathcal{E}, \mathcal{F})$ is a Dirichlet form. Its energy measure is given by $\|\nabla u\|_H^2 d\mu$ on E . However, as the Gaussian measure μ is quasi-invariant under translations by elements of the (infinite dimensional) generalized Cameron-Martin space H , the space H has zero Gaussian measure, hence zero energy measure. In other words, the space H is too small to carry a nontrivial energy measure, but on the larger space E the energy measures generally are nontrivial.

7. SEPARATION OF POINTS AND SEPARABLE RESISTANCE FORMS

In addition to Assumption 3.1 we now assume the following.

Assumption 7.1. The space \mathcal{B} separates points, that is for each $x, y \in X$, there are $f \in \mathcal{B}$ such that $f(x) \neq f(y)$.

An immediate consequence of this assumption is that $\iota : X \rightarrow \Delta$ is injective, so X is embedded in Δ as $\iota(X)$. Thus we will use X and $\iota(X)$ interchangeably. We further assume that $\iota(X)$ is a Borel set with respect to the Gelfand topology in Δ , although this assumption is technical and often can be weakened or eliminated, depending on the situation.

Remark 7.1.

- (i) Assumption 7.1 leads to a situation similar to the one in [2, Section 2]. See also the references cited there.
- (ii) If \mathcal{B} does not separate points, one can define an equivalence relation \sim on X by $x \sim y$ if $f(x) = f(y)$ for all $f \in \mathcal{B}$. Then all functions in \mathcal{B} naturally define functions on the quotient space $\tilde{X} = X/\sim$, and functions in \mathcal{B} separates equivalent classes. In this case, $\tilde{\iota} : \tilde{X} \rightarrow \Delta$, defined by $\tilde{\iota}([x]) = \iota(x)$ is an embedding with $\tilde{\iota}(\tilde{X}) = \iota(X)$.

In light of 4.1, any $\mu \in \mathcal{D}(\mathcal{B})$ can be extended to a positive measure on Δ . By Assumption 7.1 we may consider the measure of X in Δ . In particular, for any σ -finite $\mu \in \mathcal{D}(\mathcal{B})$, we can extend μ to Δ either by considering $\hat{\mu}$, or by

$$\nu(A) = \mu(A \cap X).$$

However, by equation (8) and the Riesz representation theorem, $\hat{\mu}$ and ν coincide.

The fact that X is a set of full measure $\hat{\mu}$ allows us a technique for extending results for Dirichlet forms on locally compact spaces to a more general class of spaces. The following result is a version of [2, Theorem 2.7].

Proposition 7.1. *Since $\hat{\mathcal{E}}$ is a regular Dirichlet form on Δ , there is a $\hat{\mu}$ -symmetric Hunt process on Δ with Dirichlet form $\hat{\mathcal{E}}$. Since $A(\mathcal{B}(\mathcal{E}))$ separates points, X is naturally identified as a subset of Δ with full $\hat{\mu}$ -measure. By [14, Lemma 4.1.1], this implies that the process on Δ is contained in X with probability 1, thus can be thought of as a process on X .*

Note that we do not claim that this process is a Hunt process on X because we do not consider X as a topological space. However the random process is well defined, which is useful in some applications such as the following.

In what follows we will consider a special class of Dirichlet forms, the resistance forms of Kigami [20, 21, 22], for which points have positive capacity. For simplicity we define these forms in the separable case, which can be essentially reduced to a form on a countable set.

Definition 7.1. A pair $(\mathcal{E}, \mathcal{F})$ is called a resistance form on a countable set V_* if it satisfies:

- (RF1) \mathcal{F} is a linear subspace of the functions $V_* \rightarrow \mathbb{R}$ that contains the constants, \mathcal{E} is a nonnegative symmetric quadratic form on \mathcal{F} , and $\mathcal{E}(u, u) = 0$ if and only if u is constant.
- (RF2) The quotient of \mathcal{F} by constant functions is Hilbert space with the norm $\mathcal{E}(u, u)^{1/2}$.
- (RF3) If v is a function on a finite set $V \subset V_*$ then there is $u \in \mathcal{F}$ with $u|_V = v$.
- (RF4) For any $x, y \in V_*$ the effective resistance between x and y is
$$R(x, y) = \sup \left\{ \frac{(u(x) - u(y))^2}{\mathcal{E}(u, u)} : u \in \mathcal{F}, \mathcal{E}(u, u) > 0 \right\} < \infty.$$
- (RF5) (Markov Property.) If $u \in \mathcal{F}$ then $\bar{u}(x) = \max(0, \min(1, u(x))) \in \mathcal{F}$ and $\mathcal{E}(\bar{u}, \bar{u}) \leq \mathcal{E}(u, u)$.

The resistance forms on countable sets are determined by a sequence of traces on finite subsets, as in the following two propositions.

Proposition 7.2 ([20, 21, 22]). *Resistance forms have the following properties.*

- (i) $R(x, y)$ is a metric on V_* . Functions in \mathcal{F} are R -continuous, thus have unique R -continuous extension to the R -completion X_R of V_* .
- (ii) If $U \subset V_*$ is finite then a Dirichlet form \mathcal{E}_U on U may be defined by

$$\mathcal{E}_U(f, f) = \inf \{ \mathcal{E}(g, g) : g \in \mathcal{F}, g|_U = f \}$$

in which the infimum is achieved at a unique g . The form \mathcal{E}_U is called the trace of \mathcal{E} on U , denoted $\mathcal{E}_U = \text{Trace}U(\mathcal{E})$. If $U_1 \subset U_2$ then $\mathcal{E}_{U_1} = \text{Trace}U_1(\mathcal{E}_{U_2})$.

Proposition 7.3 ([20, 21, 22]). *Suppose $V_n \subset V_*$ are finite sets such that $V_n \subset V_{n+1}$ and $\bigcup_{n=0}^{\infty} V_n$ is R -dense in V_* . Then $\mathcal{E}_{V_n}(f, f)$ is non-decreasing and $\mathcal{E}(f, f) = \lim_{n \rightarrow \infty} \mathcal{E}_{V_n}(f, f)$ for any $f \in \mathcal{F}$. Hence \mathcal{E} is uniquely defined by the sequence of finite dimensional traces \mathcal{E}_{V_n} on V_n .*

Conversely, suppose V_n is an increasing sequence of finite sets each supporting a resistance form \mathcal{E}_{V_n} , and the sequence is compatible in that each \mathcal{E}_{V_n} is the trace of $\mathcal{E}_{V_{n+1}}$ on V_n . Then there is a resistance form \mathcal{E} on $V_ = \bigcup_{n=0}^{\infty} V_n$ such that $\mathcal{E}(f, f) = \lim_{n \rightarrow \infty} \mathcal{E}_{V_n}(f, f)$ for any $f \in \mathcal{F}$.*

The following theorem follows easily from the analysis presented above. See [Chapter 5][22] for discussion why the effective resistance metric is not suitable to define topology to produce a regular Dirichlet form.

Theorem 7.1. *There exists a finite measure μ on $X = V_*$ such that:*

- (i) *any point of X has positive measure and any function of finite energy is in $L^2(X, \mu)$;*
- (ii) *\mathcal{E} is a Dirichlet form on $L^2(X, \mu)$;*
- (iii) *the embedding of X into the Gelfand spectrum Δ yields a regular Dirichlet form on $L^2(\Delta, \mu)$.*

Moreover, one can see that for any other finite measure μ on V_* one can obtain a regular Dirichlet form on $L^2(\Delta, \mu)$ by modifying the domain. However the case of infinite measures is more delicate. For instance, in [Chapter 5][22] one can see that choosing the counting measure on V_* may not produce a regular Dirichlet form, even though the space X is compact in the topology induced by the set of functions of finite energy (but is not locally compact in the topology induced by the effective resistance metric).

REFERENCES

- [1] S. Albeverio, Z.-M. Ma, M. Röckner, *A Beurling-Deny type structure theorem for Dirichlet forms on general state spaces.* — in: Ideas and Methods in Mathematical Analysis, Stochastics and Applications, 115-123 Ed. S. Albeverio, J.E. Fenstad, H. Holden, T. Lindström, Cambridge Univ. Press, Cambridge 1992.
- [2] S. Albeverio, M. Röckner, *Classical Dirichlet forms on topological vector spaces - construction of an associated diffusion process.* — Probab. Th. Rel. Fields **83** (1989), 405-434.
- [3] S. Albeverio, M. Röckner, *Classical Dirichlet forms on topological vector spaces - closability and a Cameron-Martin formula.* — J. Funct. Anal. **88** (1990), 395-436.

- [4] G. Allain, *Sur la représentation des formes de Dirichlet*. — Ann. Inst. Fourier **25** (1975), 1-10.
- [5] W. Arveson, *An Invitation to C^* -Algebras*. Springer Graduate Texts in Math. **39**, Springer, New York, 1976.
- [6] A. Beurling, J. Deny, *Espaces de Dirichlet I, le cas élémentaire*. — Acta Math. **99** (1958), 203-224.
- [7] B. Blackadar, *Operator Algebras: Theory of C^* -Algebras and von Neumann Algebras*. Encyclopedia of Math. Sciences **122**, Springer, New York, 2006.
- [8] N. Bouleau, F. Hirsch, *Dirichlet Forms and Analysis on Wiener Space*. deGruyter Studies in Math. **14**, deGruyter, Berlin, 1991.
- [9] J. Deny, *Méthodes Hilbertiennes et théorie du potentiel*. CIME, Rome, 1970.
- [10] E.B. Dynkin, *Foundations of the Theory of Markov Processes*. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow 1959, English translation *Theory of Markov processes*. Prentice-Hall and Pergamon Press, 1965, Dover Pub., 2006.
- [11] E.B. Dynkin, *Markov Processes*. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow 1963, English translation Academic Press and Springer-Verlag, 1965.
- [12] R.M. Dudley, *Real Analysis and Probability*. Cambridge studies in adv. math. **74**, Cambridge Univ. Press, Cambridge, 2002.
- [13] P.J. Fitzsimmons, *Markov processes and nonsymmetric Dirichlet forms without regularity*. — J. Funct. Anal. **85** (1989), 287-306.
- [14] M. Fukushima, Y. Oshima and M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*. deGruyter, Berlin, New York, 1994.
- [15] B. Fuchssteiner, *When does the Riesz representation theorem hold ?* — Arch. Math. **28** (1977), 173-181.
- [16] L. Gross *Potential theory on Hilbert space*. — J. Functional Analysis **1** (1967) 123-181.
- [17] L. Gross *Abstract Wiener spaces*. — 1967 Proc. Fifth Berkeley Sympos. Math. Statist. and Probability (Berkeley, Calif., 1965/66), Vol. II: Contributions to Probability Theory, Part 1 pp. 31-42 Univ. California Press, Berkeley, Calif.
- [18] I.A. Ibragimov, Y.A. Rozanov, *Gaussian Random Processes*. Nauka, Moscow, 1970 and Springer, New York - Berlin, 1978.
- [19] E. Kaniuth, *A Course in Commutative Banach Algebras*. Springer, New York, 2009.
- [20] J. Kigami, *Analysis on Fractals*. Cambridge Tracts in Mathematics **143**, Cambridge University Press, 2001.
- [21] J. Kigami, *Harmonic analysis for resistance forms*. — J. Funct. Anal. **204** (2003), 525-544.
- [22] J. Kigami, *Resistance forms, quasisymmetric maps and heat kernel estimates*. Mem. Amer. Math. Soc. **216** (2012).
- [23] A. A. Kirillov, *A tale of two fractals*. (in Russian) MCMNO 2010.
- [24] Y. LeJan, *Mesures associées à une forme de Dirichlet. Applications*. — Bull. Soc. Math. France **106** (1978), 61-112.
- [25] Z.-M. Ma, M. Röckner, *Introduction to the Theory of Non-Symmetric Dirichlet Forms*, Universitext, Springer, Berlin, 1992.
- [26] M. Reed, B. Simon, *Methods of Modern Mathematical Physics. I. Functional Analysis*. Academic Press, 1980.
- [27] L.C.G. Rogers, D. Williams, *Diffusions, Markov Processes, and Martingales. Volume one: Foundations*, 2nd ed. Wiley, 1994.

- [28] R. S. Strichartz, *Differential Equations on Fractals: A Tutorial*. Princeton University Press, 2006.
- [29] V. N. Sudakov, *Geometric problems of the theory of infinite-dimensional probability distributions*. Trudy Mat. Inst. Steklov. **141** (1976), 191 pp. Proc. Steklov Inst. Math., Springer (1979), 178pp.

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